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Period Doubling Bifurcations for Families of Maps on $\mathbb{R}^n$

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Infinite sequences of period doubling bifurcations in one-parameter families (1-pf) of maps enjoy very strong universality properties: This is known numerically in a multitude of cases and has been shown rigorously for certain 1-pf of maps on the interval. These bifurcations occur in 1-pf of analytic maps at values of the parameter tending to a limit with the asymptotically geometric ratio $1/4.6692\ldots$. In this paper we indicate the main steps of a proof that the same is true for 1-pf of analytic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, whose restriction to $\mathbb{R}^n$ is real.

KEY WORDS: Subharmonic bifurcations; hydrodynamics; analytic maps.

1. MOTIVATION

Physical models of hydrodynamics or of other dissipative systems tend to be very complicated. In addition, the laws describing such systems are only known approximately. One is thus faced with the problem of isolating and if possible answering new types of questions which are more or less independent of a detailed knowledge of the dynamics of any given physical system. Such questions then have answers which are *universal.* A well-known field where universal answers have been obtained is the renormalization group analysis of critical phenomena in statistical mechanics.

A new universal property has been discovered by Feigenbaum[(4)] for families of maps of the interval to itself, which depend on a parameter. It states that if such families present subharmonic bifurcations, then one can

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expect an infinite cascade of such bifurcations, as the parameter is varied, and they accumulate in a fashion independent of the detailed structure of the one-parameter family of maps. Soon afterwards, several authors[3,6,7] noted in numerical experiments that this phenomenon is not restricted to maps of the interval, but that it also occurs in discrete or continuous time approximations to the kind of equations one encounters in hydrodynamics. The purpose of this paper is to show why this is true in the discrete time case.

The continuous time case is indirectly also covered by our results because there are many situations in which the continuous evolution can be described in terms of a discrete map by use of a Poincaré map. Thus our theorems below are directly relevant to hydrodynamical equations in finitely many variables, and they show that for many of these one should observe the universally scaled accumulation of period doubling bifurcations. For the Bénard flow in liquid helium, Libchaber and Maurer[9] have measured a power spectrum for such cascades which (at least qualitatively) looks similar to the power spectrum which can be predicted from the general theory developed in this paper. Feigenbaum[5] gives a heuristic derivation for this spectrum, which is based on our results. Thus we have a first indication of a possible link between the abstract theory presented here and experiments.

2. INTRODUCTION

Universal properties of one-parameter families of maps on an interval were discovered numerically by Feigenbaum[4] and investigated from a rigorous point of view in Ref. 2. In that paper the authors considered one-parameter families of transformations of the interval $[-1,1]$ into itself of the form

$$x \mapsto g_\mu(|x|^{1+\epsilon})$$

Here, μ is the parameter, $g_\mu(0)=1$, $\epsilon>0$ is small, and $g_\mu(\cdot)$ is analytic in some neighborhood of $[-1,1]$ and satisfies some other technical conditions on the μ dependence. Assuming moreover that g_μ is near to a specific function f, the following result was found.

Theorem I.[2] There is a manifold $\mathscr{W}_s$ (of codimension one in a space of analytic functions) such that the following is true. If the family $\mu \mapsto g_\mu$ is transverse to $\mathscr{W}_s$ and if ϵ is sufficiently small, then

(1) The family $\mu \mapsto g_\mu$ has infinitely many bifurcation points. These points correspond to successive bifurcations from a stable period 2^n to a stable period 2^{n+1}.

(2) If $\{\mu_i\}_{i\in\mathbb{N}}$ is the sequence of values of μ for which a period described in (1) appears, then

$$\lim_{n\to\infty}\frac{\log|\mu_\infty-\mu_n|}{n}=-\log\delta$$

where δ is a universal number, which depends only on ϵ, but not on $\mu\mapsto g_\mu$.

The most interesting case $\epsilon=1$ is not covered by this theorem but has been numerically studied by Feigenbaum.[4] In particular, he computed the value of the universal number: $\delta=4.669\ldots$.

Infinite sequences of period doubling bifurcations have also been observed in higher-dimensional systems. One of them is the Hénon map in $\mathbb{R}^2$:

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}1-\mu x^2+y\\bx\end{pmatrix}$$

For $b=0.3$, the first 11 values of μ for which a doubling bifurcation occurs were computed with good accuracy in Ref. 3. They appear to satisfy the universal behavior described in Theorem I with the same number δ. Other examples of this behavior in higher-dimensional flows were described in Refs. 6 and 7; cf. also Ref. 1.

The aim of the present paper is to outline a proof of the universal behavior for maps in finite-dimensional spaces. We shall assume that the results proven in Ref. 2 for small ϵ extend to $\epsilon=1$. Lanford reported recently on some decisive progress in this direction,[8] and our hypotheses are inspired by his results. See also Ref. 10.

Our argument is organized as follows. We first state some hypotheses on the one-dimensional case for $\epsilon=1$. We then explain how the renormalization group program can be realized for certain maps on $\mathbb{C}^n$, $n>1$. This includes the search for a fixed point of a nonlinear transformation and the study of its linearization at this fixed point. A second part of the argument should include a more detailed description of the stable and unstable manifolds $\mathcal{W}_s$ and $\mathcal{W}_u$ and of their intersection with certain submanifolds of codimension one. We have not worked out the tedious functional analytic details associated with this part of the argument. They should be similar to those of Ref. 2, and the reader is referred to that paper for an explanation of the geometrical ideas of the method. *Throughout this paper maps* $\mathbb{C}^n\to\mathbb{C}^n$ *are implicitly considered to be real on* $\mathbb{R}^n$.

The main result of the present paper is the following theorem.

Theorem II. There is a map Φ from $\mathbb{C}^n$ to $\mathbb{C}^n$, and a submanifold $\mathcal{W}_s$ in the space of analytic functions on $\mathbb{C}^n$ to $\mathbb{C}^n$ (of codimension one and passing through Φ) such that the following is true:

(1) Every once continuously differentiable one-parameter family $\mu \mapsto G_\mu$ of analytic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ which crosses transversally through $\mathcal{W}_s$ near Φ has infinitely many bifurcations from a stable period 2^m to a stable period 2^{m+1}.

(2) If $\{\mu_m\}_{m\in\mathbb{N}}$ is the sequence of values of μ for which a period 2^m described in (1) appears, then

$$\lim_{m\to\infty} \frac{\log|\mu_\infty - \mu_m|}{m} = -\log\delta$$

where $\delta = 4.669\ldots$ does not depend on the family G_μ.

3. THE ONE-DIMENSIONAL CASE. ASSUMPTIONS

Before stating our hypotheses we recall some definitions associated with the one-dimensional problem. Let $\mathfrak{M}$ be the set of functions g which map $[-1,1]$ into itself and for which $-1 < g(1) < 0$. For $g \in \mathfrak{M}$ we define $\mathfrak{T}$ by

$$\mathfrak{T}g(x) = g(1)^{-1} g \circ g(g(1)x)$$

Our first set of assumptions is the following.

(M1) The equation $\mathfrak{T}g = g$ has a solution $\phi \in \mathfrak{M}$ which is analytic in some neighborhood D_1 of $[-1,1]$.

(M2) ϕ is symmetric, $\phi(x) = f(x^2)$, and $f'(t) \neq 0$ for $t \in [0,1]$.

(M3) Define $\lambda = \phi(1)$.[3] For some positive γ,

$$2\lambda^2 \sup\{|f'(z^2)| : z \in D_1\} < 1 - \gamma < 1.$$

(M4) f has exactly one zero in $[0,1]$.

From (M1) we can investigate the derivative of $\mathfrak{T}$ at the fixed point ϕ. We obtain

$$(D\mathfrak{T}\phi h)(z) = \lambda^{-1} h(\phi(\lambda z)) + \lambda^{-1}\phi'(\phi(\lambda z))h(\lambda z)$$

Lemma 1.[4] If $\sigma(y) = y^n$ for some integer $n \geqslant 0$ then

$$\psi_\sigma(x) = -\sigma(\phi(x)) + \phi'(x)\sigma(x)$$

is an eigenvector of $D\mathfrak{T}_\phi$ with eigenvalue λ^{n-1}.

Proof. We consider the following family $S(t)$ of maps of the complex plane,

$$z \mapsto S(t)z = z + t\sigma(z)$$

[3]For the solution found by Feigenbaum and Lanford, $\phi(x) = 1 - 1.401\ldots x^2$, $\lambda = \phi(1) = -0.3995\ldots$.

which is well defined and invertible on a neighborhood of $[-1,1]$ for small t.

If M_λ denotes the operator of multiplication by λ in $\mathbb{C}$, we have

$$\begin{aligned}\mathcal{T}\big(S(t)^{-1}\circ\phi\circ S(t)\big) &= M_\lambda^{-1}\circ S(t)^{-1}\circ M_\lambda\circ\big(M_\lambda^{-1}\circ\phi\circ\phi\circ M_\lambda\big)\\ &\quad\circ M_\lambda^{-1}\circ S(t)\circ M_\lambda\\ &= \big(M_\lambda^{-1}\circ S(t)^{-1}\circ M_\lambda\big)\circ\phi\circ\big(M_\lambda^{-1}\circ S(t)\circ M_\lambda\big)\end{aligned}$$

Differentiating with respect to t and setting $t=0$ we obtain

$$D\mathcal{T}_\phi(\psi_\sigma) = \psi_{M_\lambda^{-1}\circ\sigma\circ M_\lambda}$$

by using that $\psi_\sigma = \partial_t[S(t)^{-1}\circ\phi\circ S(t)]|_{t=0}$. From this the result follows. ■

Our last set of hypotheses is the following.

(M5) The operator $D\mathcal{T}_\phi$ has a simple eigenvalue $\delta > 1$ which is different from λ^{-1}, λ^{-2}.[4] The corresponding eigenvector ρ is even. We define $r(x^2) = \rho(x)$.

(M6) The eigenvalues δ, λ^{-1}, 1 are the only eigenvalues of modulus $\geqslant 1$. Their corresponding spectral projections are one dimensional.

Lanford has essentially completed the proof of (M1), . . . , (M4).[8] His method can be extended to prove (M5) and (M6). See also Ref. 10.

Remark. In view of Lemma 1 it could seem reasonable to assume that one has found a total set of eigenvectors for $D\mathcal{T}_\phi$. This is not the case and in fact the family $\{\psi_\sigma : \sigma \text{ analytic}\}$ has infinite codimension. This can be seen as follows: For all σ, one can easily check that ψ_σ satisfies the relations

$$\begin{aligned}&\phi''(x_0)\Big[\lambda^{-j}(\phi+\epsilon\psi_\sigma)^{2^j}(\lambda^j x_0) - x_0\Big]\\ &\quad -(\phi'(x_0)-1)\Big[\frac{d}{dx}(\phi+\epsilon\psi_\sigma)^{2^j}(x)|_{x=\lambda^j x_0} - \phi'(x_0)\Big] = 0\\ &\qquad\qquad \operatorname{mod} O(\epsilon^2),\quad j=0,1,2,\ldots\end{aligned}$$

where x_0 is defined by $\phi(x_0) = x_0$. This is true because $\epsilon\psi_\sigma$ is generated through a (nonlinear) coordinate transformation, and this leaves the derivatives at the periodic points invariant. It is also easy to see that the relations for different j are independent.

[4] In fact, $\delta = 4.6692\ldots$.

4. MAPS ON $\mathbb{C}^n$

We introduce some notations for the n-dimensional problem. We will use a fixed decomposition of $\mathbb{C}^n$ into a direct sum $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1}$. If z is a vector of $\mathbb{C}^n$, its components will be written $(z_0, \mathbf{z})$. $\|\cdot\|$ will be the norm of $\mathbb{C}^n$ given by

$$\|z\| = \|\mathbf{z}\|_{\mathbb{C}^{n-1}} + |z_0|$$

where $\|\mathbf{z}\|_{\mathbb{C}^{n-1}} = (\mathbf{z}\cdot\bar{\mathbf{z}})^{1/2}$ is the usual norm in $\mathbb{C}^{n-1}$. For an open subset D of $\mathbb{C}^n$ let $\mathcal{H}(D)$ denote the space of analytic and bounded maps from D to $\mathbb{C}^n$. Equipped with the norm

$$\|h\| = \sup\{\|h(z)\| : z \in D\}$$

this space is a Banach space. We shall mostly consider $\mathcal{H}_\Delta = \mathcal{H}(D(\Delta))$, where $D(\Delta)$ is the convex set

$$D(\Delta) = \{z \in \mathbb{C}^n : \|z - (y_0, \mathbf{0})\| < \Delta \quad \text{for some } y_0 \in [-1, 1]\}$$

We now fix a nonzero vector $\boldsymbol{\alpha}$ in $\mathbb{C}^{n-1}$ whose norm is bounded by two. Φ will always denote the map

$$z \mapsto \Phi(z) = (f(\zeta(z)), \mathbf{0}) \tag{1}$$

where $\zeta(z) = z_0^2 - \boldsymbol{\alpha}\cdot\mathbf{z}$ and where f is the function described in (M2). Note that if Δ is sufficiently small, so that $\{\zeta^{1/2}(z) : z \in D(\Delta)\}$ is contained in D_1, then Φ belongs to $\mathcal{H}_\Delta$. The universal behavior asserted in Theorem II will be proven in the sequel for one-parameter families of maps which are near to Φ.

It might seem that this is an undue restriction on the one-parameter family. Note, however, that our problem is invariant under C^1 coordinate transformations. This means that given a one-parameter family $\tilde{G}_\mu$ of maps one might find transformations τ_μ such that $G_\mu = \tau_\mu^{-1} \circ \tilde{G}_\mu \circ \tau_\mu$ satisfies the conditions of Theorem II. In particular, τ_μ might be constant and change the direction or length of $\boldsymbol{\alpha}$. The conclusions of Theorem II are then valid for G_μ as well as for $\tilde{G}_\mu$.

We next define the renormalization transformation. Let Λ be the diagonal $n \times n$ matrix given by

$$\Lambda z = (\lambda z_0, \lambda^2 \mathbf{z})$$

where $\lambda = \phi(1) = -0.3995\ldots$.

Lemma 2. If Δ is sufficiently small and if G belongs to $\mathcal{H}_\Delta$, and $\|G - \Phi\| < \gamma\Delta$,[5] then $\Lambda^{-1} \circ G \circ G \circ \Lambda$ also belongs to $\mathcal{H}_\Delta$.

[5] Cf. (M3) for the definition of γ.

Proof. From the hypotheses on f and on $u = G - \Phi$ it is easy to verify, using (M3), that for $z \in D(\Delta)$,

$$G(\Lambda z) = \big(f\big(\lambda^2\zeta(z)\big) + u_0(\Lambda z), \mathbf{u}(\Lambda z)\big)$$

is again in $D(\Delta)$, and therefore $\Lambda^{-1} \circ G \circ G(\Lambda z)$ is well defined. The analyticity follows from that of G and of $G(\Lambda \cdot)$. ■

Definition. The transformation $\mathcal{N}$ given by

$$\mathcal{N} : G \mapsto \Lambda^{-1} \circ G \circ G \circ \Lambda$$

will be called the renormalization transformation.

Owing to Lemma 2, this transformation maps the ball in $\mathcal{H}_\Delta$ centered at Φ and of radius $\gamma\Delta$ into $\mathcal{H}_\Delta$.

Lemma 3. Φ is a fixed point of $\mathcal{N}$.

Proof. This is an easy consequence of (M1), (M2), and of the definition of $\mathcal{N}$ and Φ. ■

According to the general philosophy of the renormalization group analysis, we shall now investigate the spectrum of the derivative of $\mathcal{N}$ at Φ. One can easily derive the following expression for the derivative $D\mathcal{N}_G$, where $G \in \mathcal{H}_\Delta$ and $\|G - \Phi\| < \Delta\gamma/2$:

$$(D\mathcal{N}_G h)(z) = \Lambda^{-1}\big[h\big(G(\Lambda z)\big) + DG_{G(\Lambda z)} h(\Lambda z)\big]$$

Proposition 4. For sufficiently small Δ, the following assertions hold.

(1) $\mathcal{N}$ is a C^2 transformation on $\mathcal{H}_\Delta$ defined in a ball centered at Φ and with radius $\gamma\Delta/2$. $\|D^2\mathcal{N}_{\Phi+u}(h,k)\| \leqslant O(1)\|h\|\,\|k\|$ provided $\|u\| < \gamma\Delta/2$.

(2) $D\mathcal{N}_\Phi$ is a compact operator from $\mathcal{H}_\Delta$ into itself.

Proof. We remark that for $\|u\| < \gamma\Delta/2$ and sufficiently small Δ, $(\Phi + u)$ maps the closure of $\Lambda D(\Delta)$ into $D(\Delta)$ by (M1) and (M3). The assertion (1) is now verified by a direct computation. The compactness of $D\mathcal{N}_\Phi$ is a consequence of Montel's theorem. ■

We start now an analysis of the spectrum of $D\mathcal{N}_\Phi$. Our result is

Theorem 5

(1) If $\sigma = (\sigma_0, \boldsymbol{\sigma})$ is an analytic map from $\mathbb{C}^n$ to $\mathbb{C}^n$, and if $\Lambda^{-1} \circ \sigma \circ \Lambda = \lambda^m \sigma$ for some integer $m \geqslant -2$, then the map Ψ_σ defined by

$$\Psi_\sigma(z) = -\big(\sigma_0(f(\zeta(z)), \mathbf{0}), \boldsymbol{\sigma}(f(\zeta(z)), \mathbf{0})\big) + \big(2z_0 f'(\zeta(z))\sigma_0(z) - f'(\zeta(z))\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}(z), \mathbf{0}\big)$$

is an eigenvector of $D\mathcal{N}_\Phi$ with eigenvalue λ^m.

(2) δ, λ^{-2}, λ^{-1}, and 1 are the only eigenvalues of $D\mathcal{N}_\Phi$ of modulus greater or equal to 1. The corresponding spectral subspaces are spanned (i) for δ by $P = (r \circ \zeta, \mathbf{0})$, where r is the function defined in (M5), and (ii) for λ^{-2}, λ^{-1}, 1 by the vectors Ψ_σ with $\Lambda^{-1} \circ \sigma \circ \Lambda$ equal to $\lambda^{-2}\sigma$, $\lambda^{-1}\sigma$, and σ, respectively. They have the dimensions 1, $n-1$, n, and $n^2 - n + 1$.

Proof

(1) This can be shown by a method analogous to that of Lemma 1.

(2) It is sufficient to prove the assertions for the restrictions of $D\mathcal{N}_\Phi$ to a closed linear subspace which contains $D\mathcal{N}_\Phi\mathcal{H}_\Delta$. The direct sum $\hat{\mathcal{H}}(\Delta) = \mathcal{H}_0(\Delta) \oplus \mathcal{H}(\Delta)$ has this property, where

$$\mathcal{H}_0(\Delta) = \{h : h = (h_0, \mathbf{0}) \in \mathcal{H}_\Delta\}$$

$$\mathbf{H}(\Delta) = \{h : h(z) = (0, \mathbf{h}_1(\zeta(z))) \text{ with } \mathbf{h}_1 \text{ analytic and bounded on } D_0(\Delta)\}$$

where

$$D_0(\Delta) = \{\zeta(z) : z \in D(\Delta)\}$$

The restriction of $D\mathcal{N}_\Phi$ to $\hat{\mathcal{H}}(\Delta)$ will be denoted by A. Let $\mathcal{H}' = \mathcal{H}(\Phi(D(\Delta)) \cup D(\Delta))$. The following two lemmas will be proven below.

Lemma 6. Let $(A-\mu)^\nu \Psi_\tau = 0$ for some $\tau \in \mathcal{H}'$ and some $\nu \in \mathbb{N}$. If $\mu \in \{\lambda^k : k = -2, -1, 0, \ldots\}$ then $\Psi_\tau = \Psi_\sigma$ for some σ which is analytic in $\mathbb{C}^n$ and for which $\Lambda^{-1} \circ \sigma \circ \Lambda = \mu\sigma$. Otherwise $\Psi_\tau = 0$.

Lemma 7. Let $u \in \hat{\mathcal{H}}(\Delta)$ and let $(A-\mu)^\nu u = 0$ for some μ with $|\mu| \geqslant 1$. Then $u = cP + \Psi_\tau$ for some $c \in \mathbb{C}$ and some $\tau \in \mathcal{H}'$.

We complete the proof of Theorem 5, part (2). Let $u \in \hat{\mathcal{H}}(\Delta)$ and in a spectral subspace of $D\mathcal{N}_\Phi$ with eigenvalue μ, $|\mu| \geqslant 1$. Since $D\mathcal{N}_\Phi$ is compact, and $\mu \neq 0$, this means that for some $\nu \in \mathbb{N}$, $(A-\mu)^\nu u = 0$. By Lemma 7, we conclude that for some c, $u = cP + \Psi_\tau$ for some $\tau \in \mathcal{H}'$. Suppose $\mu = \delta$. Then $(A-\mu)^\nu cP = 0$ and hence $(A-\mu)^\nu \Psi_\tau = 0$, so that by Lemma 6, $\Psi_\tau = 0$. If $\mu \neq \delta$ then $-cP = (\delta-\mu)^{-\nu}(A-\mu)^\nu \Psi_\tau = \Psi_\kappa$ for some $\kappa \in \mathcal{H}'$. From $(A-\delta)^\nu cP = 0$ we have thus $(A-\delta)^\nu \Psi_\tau = 0$ and applying Lemma 6 again we must either have $\Psi_\tau = 0$ or $\mu = \lambda^k$, $k \in \{-2, -1, 0\}$ and $u = \Psi_\tau = \Psi_\sigma$ for a σ with $\Lambda^{-1} \circ \sigma \circ \Lambda = \lambda^k$. This completes the proof of Theorem 5. ■

Proof of Lemma 6. For $\epsilon \neq 0$ let $N(\epsilon) = \{z \in \mathbb{C}^n : \|z\| < |\epsilon|\}$. We define the bounded operators[6]

$$\hat{A} : \mathcal{H}(N(\epsilon)) \to \mathcal{H}(N(\lambda^{-1}\epsilon)) \qquad \text{by } \hat{A}\sigma = \Lambda^{-1} \circ \sigma \circ \Lambda$$

[6] To shorten the notation we omit the ϵ dependence of $\hat{A}$. This can be rendered completely rigorous by writing the proper injections, etc.

and for $|\lambda\eta| < 1$,

$$\hat{U}(\eta) : \mathcal{H}(N(\epsilon)) \to \mathcal{H}(N(\lambda^2\epsilon))$$

$$\text{by } \big(\hat{U}(\eta)\sigma\big)(z) = \big(\sigma_0(\eta^2 z_0, \eta\mathbf{z}), \eta\boldsymbol{\sigma}(\eta^2 z_0, \eta\mathbf{z})\big)$$

One can easily check that

$$\hat{U}(\eta)\hat{A} = \hat{A}\hat{U}(\eta) = \lambda^{-2}\hat{U}(\lambda\eta)$$

By defining $U(\eta)\Psi_\kappa = \Psi_{\hat{U}(\eta)\kappa}$ for $\kappa \in \mathcal{H}(N(\lambda^{-3}))$, we obtain the corresponding relation

$$U(\eta)A\Psi_\kappa = AU(\eta)\Psi_\kappa = \lambda^{-2}U(\lambda\eta)\Psi_\kappa$$

where we have used that $A\Psi_\kappa = \Psi_{\hat{A}\kappa}$.

Let us now assume that $(A-\mu)^\nu\Psi_\tau = 0$ for some integer $\nu \geqslant 1$ and some $\tau \in \mathcal{H}'$. By choosing $M \in \mathbb{N}$ sufficiently large, we can achieve that $\sigma = \hat{A}^M\hat{D}^M\tau$, where $\hat{D} = -\sum_{j=1}^{\nu}\binom{\nu}{j}\hat{A}^{j-1}(-\mu)^{-j}$, belongs to $\mathcal{H}(N(\lambda^{-3}))$. Moreover, $\Psi_\tau = \Psi_\sigma$, and

$$\eta \mapsto U(\eta)\Psi_\sigma$$

defines an analytic family of maps in $\mathcal{H}_\Delta$ for $|\lambda\eta| < 1$. Thus in a disc of radius larger than one the following series converges:

$$U(\eta)\Psi_\sigma = \sum_{k=0}^{\infty}\eta^k\Psi_{\sigma_{0,k}}$$

where $\sigma_{\eta,k} = (1/k!)\partial_\eta^k\hat{U}(\eta)\sigma$. This is an expansion into eigenvectors of $D\mathfrak{N}_\Phi$ since

$$\begin{aligned}\hat{A}\sigma_{0,k} &= \frac{1}{k!}\partial_\eta^k\hat{A}\hat{U}(\eta)\sigma|_{\eta=0} = \lambda^{-2}\frac{1}{k!}\partial_\eta^k\hat{U}(\lambda\eta)\sigma|_{\eta=0}\\ &= \frac{\lambda^{k-2}}{k!}\partial_\eta^k\hat{U}(\eta)\sigma = \lambda^{k-2}\sigma_{0,k}\end{aligned}$$

For all η with $|\eta\lambda| < 1$ one has

$$0 = U(\eta)(A-\mu)^\nu\Psi_\sigma = (A-\mu)^\nu U(\eta)\Psi_\sigma$$

This implies that for $k = 0, 1, 2, \ldots$

$$(A-\mu)^\nu\Psi_{\sigma_{0,k}} = 0$$

But $(A-\mu)^\nu\Psi_{\sigma_{0,k}} = (\lambda^{k-2}-\mu)^\nu\Psi_{\sigma_{0,k}}$, which leads to the conclusion that either $\Psi_{\sigma_{0,k}} = 0$ or $\mu = \lambda^{k-2}$. The assertion follows by using that

$$\Psi_\tau = U(1)\Psi_\sigma = \sum_{k=0}^{\infty}\Psi_{\sigma_{0,k}} \qquad ■$$

Proof of Lemma 7. It will be useful to construct a partition of the operator 1 on $\hat{\mathcal{H}}(\Delta)$ which commutes with A. For a map $h = (h_0, \mathbf{h}_1 \circ \zeta)$ in

$\hat{\mathcal{H}}(\Delta)$ we define

$$\begin{aligned} \boldsymbol{\nu} &= \mathbf{h}_1\big(f^{-1}(0)\big) \\ \mathbf{h}_2(\zeta) &= (\mathbf{h}_1(\zeta) - \boldsymbol{\nu})/f(\zeta) \end{aligned} \tag{2}$$

and

$$\begin{aligned} (\Pi_0 h)(z) &= \big(h_0(z) - f'(\zeta(z))\boldsymbol{\alpha}\cdot\boldsymbol{\nu} - \tfrac{1}{2}\boldsymbol{\alpha}\cdot\mathbf{h}_2(\zeta(z)), \mathbf{0}\big) \\ (\Pi_1 h)(z) &= \big(f'(\zeta(z))\boldsymbol{\alpha}\cdot\boldsymbol{\nu}, \boldsymbol{\nu}\big) \qquad \{= \Psi_{(0,-\boldsymbol{\nu})}\} \\ (\Pi_2 h)(z) &= \big(\tfrac{1}{2}\boldsymbol{\alpha}\cdot\mathbf{h}_2(\zeta(z)), f(\zeta(z))\mathbf{h}_2(\zeta(z))\big) \end{aligned}$$

This defines linear operators Π_0, Π_1, Π_2 on $\hat{\mathcal{H}}(\Delta)$, which by our hypotheses on f are bounded by $O(\Delta^{-1})$. From the explicit form of A and the fact that $f(\lambda^2 f^{-1}(0))^2 = f^{-1}(0)$, one easily obtains the relations

$$\begin{aligned} \Pi_0 + \Pi_1 + \Pi_2 &= 1 \\ \Pi_i \Pi_j &= \delta_{ij}\Pi_j, \qquad i,j = 0,1,2 \\ \Pi_i A &= A\Pi_i, \qquad i = 0,1,2 \end{aligned} \tag{3}$$

In view of (3) it is sufficient to show that the hypotheses of the lemma imply $\Pi_i u \sim cP$ for some $c \in \mathbb{C}$, where $h \sim k$ means $h - k \in \{\Psi_\sigma : \sigma \in \mathcal{H}'\}$.

The case $i = 1$ follows since for all $h \in \hat{\mathcal{H}}(\Delta)$, $\Pi_1 h$ is an eigenvector of A of the form Ψ_σ with eigenvalue λ^{-2}, and thus $\Pi_1 u \sim 0$.

Next we will show that $\Pi_2 u \sim 0$. For maps h in $\mathcal{H}_2(\Delta) = \Pi_2\hat{\mathcal{H}}(\Delta)$ we define

$$\mathbf{B}(h) = \mathbf{h}_2 \circ f^{-1}$$

where $\mathbf{h}_2$ is defined in Eq. (2).

Since, by (M2), $m^{-1} < |f'(\zeta)| < m$ for some $m \in \mathbb{N}$ and for all $\zeta \in D_0(\Delta)$, the components of $\mathbf{B}(h)$ are well defined as analytic and bounded functions on $f(D_0(\Delta))$, which contains an open neighborhood of zero. It is easily seen that the action of A on $\mathcal{H}_2(\Delta)$ induces

$$\mathbf{B}(Ah)(z) = \lambda^{-1}\mathbf{B}(h)(\lambda z)$$

i.e., it enlarges the domain of analyticity. Since Π_i commutes with A, we find from $(A - \mu)^\nu u = 0$ that

$$\Pi_i u = -\sum_{k=1}^{\nu}\binom{\nu}{k}(-\mu)^{-k}A^k\Pi_i u, \qquad i = 0,1,2 \tag{4}$$

This implies that $\mathbf{B}(\Pi_2 u)$ is analytic on $\mathbb{C}$. Furthermore $\Pi_2 u = \Psi_\sigma$ with

$$\sigma(z) = -\big(2\boldsymbol{\alpha}\cdot\mathbf{B}(\Pi_2 u)(z_0), z_0\mathbf{B}(\Pi_2 u)(z_0)\big)$$

and thus $\Pi_2 u \sim 0$.

Finally we will show that $\Pi_0 u \sim cP$ for some $c \in \mathbb{C}$. Define the bounded operators $S:\mathcal{H}_0(\Delta)\to\mathcal{H}(D_0(\Delta))$ and $S^*:\mathcal{H}(D_0(\Delta))\to\mathcal{H}_0(\Delta)$ by

$$(Sh)(z_0)=h_0(z_0,\mathbf{0})$$
$$(S^*k)(z_0,\mathbf{z})=(k(z_0),\mathbf{0})$$

and set $R=S^*S$. These operators have the following properties:

$$SS^*=Id_{\mathcal{H}_0(\Delta)}$$
$$RA(1-R)|_{\mathcal{H}_0(\Delta)}=0$$
$$SAS^*=D\mathcal{T}_\phi \qquad \text{(the linearized operator for one dimension)}$$

Now let $C=RAR$. Since $CS^*=S^*D\mathcal{T}_\phi$ and $C(1-R)=0$, the spectral subspace $\mathcal{H}$ corresponding to eigenvalues of modulus greater than or equal to 1 of C is spanned by the eigenvectors $S^*\rho$, $S^*\psi_{S\kappa}$, and $S^*\psi_{S\tau}$, where $\kappa(z)=(1,\mathbf{0})$ and $\tau(z)=(z_0,\mathbf{0})$. They differ from P, Ψ_κ, and Ψ_τ (these are eigenvectors of A with corresponding eigenvalues) only by elements in $\mathcal{H}_0(\Delta)$ which map vectors $(z_0,\mathbf{0})$ to zero. Since for Δ sufficiently small the functions f, f', and r are analytic and bounded on $\zeta(\Phi(D(\Delta))\cup D(\Delta))$, these differences can be written in the form

$$\Psi_\sigma(z)=-(f'(\zeta(z))\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}(z),\mathbf{0}) \tag{5}$$

with $\sigma=(0,\boldsymbol{\sigma})$, $\boldsymbol{\sigma}(z_0,\mathbf{0})=0$, and $\Psi_\sigma\sim 0$. In other words,

$$S^*\rho\sim P$$
$$S^*\psi_{S\kappa}\sim\Psi_\kappa\sim 0$$
$$S^*\psi_{S\tau}\sim\Psi_\tau\sim 0$$

The assertion $\Pi_0 u\sim cP$ will now be proven by showing that $\Pi_0 u\sim k$ for some $k\in\mathcal{H}$.

By using that $RA(1-R)=0$ we obtain

$$\begin{aligned}0&=(A-\mu)^\nu\Pi_0 u\\&=(C-\mu)^\nu\Pi_0 u+(-\mu)^\nu v\\&=(C-\mu)^\nu(\Pi_0 u+v)\end{aligned}$$

where

$$v=(1-R)A\sum_{k=1}^{\nu}(-\mu)^{-k}(A-\mu)^{k-1}\Pi_0 u$$

This implies that $\Pi_0 u+v\in\mathcal{H}$. From (4) it follows that for every $k\in\mathbb{N}$ there is a $w_k\in\mathcal{H}_0(\Delta)$ such that $v=(1-R)A^k w_k$. The operator A substitutes $z\overset{X_1}{\mapsto}f(\lambda^2\zeta(z))$ or $z\overset{X_2}{\mapsto}\Lambda z$ in the argument of the function on which it acts. By (M3), for sufficiently large k, $\prod_{i=1}^{k}X_{j_i}$ and $\prod_{i=1}^{k}X_{j_i}\Phi$ are contractions on $D(\Delta)$. It follows that v is analytic and bounded on

$\Phi(D(\Delta)) \cup D(\Delta)$, and since $v(z_0, \mathbf{0}) = 0$, it is of the form (5), and therefore we have $v \sim 0$. This concludes the proof of Lemma 7. ∎

5. THE DEFINITION OF A RENORMALIZATION TRANSFORMATION

In the usual renormalization group analysis, the derivative of the renormalization transformation at the fixed point has only one eigenvalue greater than or equal to 1. We shall now show that the eigenvalues λ^{-2}, λ^{-1}, and 1 of $D\mathcal{N}_\Phi$ can be removed by the appropriate choice of a new renormalization transformation T.

From the definition of Ψ_σ,

$$\Psi_\sigma = \partial_t (I + t\sigma)^{-1} \circ \Phi \circ (I + t\sigma)\big|_{t=0}$$

it can be seen that the eigenvalues λ^n, $n = -2, -1, 0$, correspond to degrees of freedom associated to some change of coordinates (in particular the eigenvalue 1 corresponds to transformations which are compatible with our choice of the z_0 axis, i.e., which commute with Λ). Since we intend to describe only coordinate-independent properties, the eigenvalues λ^n can be eliminated and ultimately play no role in the universal behavior. We shall now work towards the construction of a new renormalization transformation whose derivative at the fixed point has spectrum inside the unit circle except for δ.

Let E denote the spectral projection of $D\mathcal{N}_\Phi$ associated to the eigenvalues λ^{-2}, λ^{-1}, 1. The first step is the definition of a map $h \mapsto \sigma[h]$ which satisfies

$$\Psi_{\sigma[h]} = Eh$$

Proposition 8. Define $D' = \Phi(D(\Delta)) \cup D(\Delta)$. For any h in $\mathcal{H}_\Delta$, the equation

$$\Psi_{\sigma[h]} = Eh$$

has a unique solution $\sigma[h]$ in $\mathcal{H}(D')$. The map $h \mapsto \sigma[h]$ is linear and bounded.

Proof. Let $\mathcal{K}$ be the following finite-dimensional subspace of $\mathcal{H}(D')$:

$$\mathcal{K} = \Big\{ \sigma : \sigma(z) = \nu + z_0 \nu' + \big(0, z_0^2 \boldsymbol{\mu} + \boldsymbol{\mu}'(\mathbf{z})\big) \text{ with } \nu, \nu' \in \mathbb{C}^n, \boldsymbol{\mu} \in \mathbb{C}^{n-1}$$
$$\text{and } \boldsymbol{\mu}' \text{ a linear operator from } \mathbb{C}^{n-1} \text{ into itself} \Big\}$$

It is easy to verify that $\sigma \mapsto Q\sigma = \Psi_\sigma$ is a bounded linear operator from $\mathcal{K}$ to $E\mathcal{H}_\Delta$. By Theorem 5, part 2, we have $\dim Q\mathcal{K} = \dim E\mathcal{H}_\Delta$. Therefore Q has an inverse Q^{-1} and we can define $\sigma[h] = Q^{-1}Eh$. ∎

We are now able to define our final renormalization transformation T. The explicit expression is

$$T : h \mapsto$$
$$(I + \sigma[D\mathfrak{N}_\Phi h]) \circ \Lambda^{-1} \circ (\Phi + h) \circ (\Phi + h) \circ \Lambda \circ (I + \sigma[D\mathfrak{N}_\Phi h])^{-1} - \Phi$$

Since for sufficiently small $\|h\|$ the transformation $z \mapsto z + \sigma[D\mathfrak{N}_\Phi h](z)$ maps $D(\Delta)$ analytically and one-to-one onto some neighborhood of $D(\Delta/2)$, this transformation T is well defined in some neighborhood of zero in $\mathcal{H}_\Delta$.

The properties of T are summarized in the following theorem.

Theorem 9. If Δ is sufficiently small, then

(1) T is a C^2 transformation from a neighborhood of zero in $\mathcal{H}_\Delta$ to $\mathcal{H}_\Delta$.
(2) $DT_0 = (1 - E)D\mathfrak{N}_\Phi$, where $DT_0 = DT_G$ at $G = 0$.
(3) DT_0 is compact and its spectrum consists of the simple eigenvalue δ, and a remainder strictly inside the unit disk. The eigenvector corresponding to δ is
$$P(z) = (r(\zeta(z)), \mathbf{0})$$

Proof. The proof is an immediate consequence of our previous results. ■

The theorem above is the main ingredient for the analysis of universal behavior of maps, which now follows very closely the one given in Ref. 2. We have not, however, worked out all the details of the proofs of the steps of this construction, but we believe they should not be very different from the one-dimensional case. From the existence of the fixed point 0 for the map T and from its spectral properties one deduces the existence of stable and unstable manifolds $\mathcal{W}_s$ and $\mathcal{W}_u$ for T in a neighborhood of 0. The main point of our preceding analysis is that $\mathcal{W}_s$ will have codimension one and $\mathcal{W}_u$ will have dimension one; furthermore T is expanding by a factor δ on $\mathcal{W}_u$. Now let $\Sigma_0 = \{G - \Phi : G$ has one fixed point in $D(\Delta)$ and DG has one eigenvalue -1 at this fixed point$\}$. Let $\Sigma_m = T^{-m}(\Sigma_0)$. These manifolds are, for sufficiently large m, transversal to $\mathcal{W}_u$, and the intersection of a curve $\mu \to G_\mu$ (which is near $\mathcal{W}_u$) with Σ_m corresponds to the point of bifurcation from a stable period 2^m to a stable period 2^{m+1} for G_μ. Owing to the differentiability of T near 0, the distance of Σ_m from 0 goes as constant δ^{-m} and this is the main ingredient of the proof of Theorem II.

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